

Classification of Harish-Chandra modules over the W -algebra $W(2, 2)$ *

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Abstract

In this paper, we classify all irreducible weight modules with finite dimensional weight spaces over the W -algebra $W(2, 2)$. Meanwhile, all indecomposable modules with one dimensional weight spaces over the W -algebra $W(2, 2)$ are also determined.

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1 Introduction

The W -algebra $W(2, 2)$ was introduced in [ZD] for the study of classification of vertex operator algebras generated by weight 2 vectors. By definition, *the W -algebra $W(2, 2)$* is the Lie algebra \mathcal{L} with \mathbb{C} -basis $\{L_m, I_m, C, C_1 | m \in \mathbb{Z}\}$ subject to the following relations.

Definition 1.1. *The W -algebra $\mathcal{L} = W(2, 2)$ is a Lie algebra over \mathbb{C} (the field of complex numbers) with the basis*

$$\{x_n, I(n), C, C_1 | n \in \mathbb{Z}\}$$

and the Lie bracket given by

$$[x_n, x_m] = (m - n)x_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C, \quad (1.1)$$

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$$[x_n, I(m)] = (m - n)I(n + m) + \delta_{n, -m} \frac{n^3 - n}{12} C_1, \quad (1.2)$$

$$[I(n), I(m)] = 0, \quad (1.3)$$

$$[\mathcal{L}, C] = [\mathcal{L}, C_1] = 0. \quad (1.4)$$

The W -algebra $W(2, 2)$ can be realized from the semi-product of the Virasoro algebra Vir and the Vir-module $\mathcal{A}_{0, -1}$ of the intermediate series in [OR]. In fact, let $W = \mathbb{C}\{x_m \mid m \in \mathbb{Z}\}$ be the Witt algebra (non-central Virasoro algebra) and $V = \mathbb{C}\{I(m) \mid m \in \mathbb{Z}\}$ be a W -module with the action $x_m \cdot I(n) = (n - m)I(m + n)$, then $W(2, 2)$ is just the universal central extension of the Lie algebra $W \ltimes V$ (see [OR] and [GJP]). The W -algebra $W(2, 2)$ studied in [ZD] is the restriction for $C_1 = C$ of $W(2, 2)$ in our paper.

The W -algebra $W(2, 2)$ can be also realized from the so-called *loop-Virasoro algebra* (see [GLZ]). Let $\mathbb{C}[t, t^{-1}]$ be the Laurents polynomial ring over \mathbb{C} , then the loop-Virasoro algebra $\tilde{V}L$ is the universal central extension of the loop algebra $\text{Vir} \otimes \mathbb{C}[t^1, t^{-1}]$ and $W(2, 2) = \tilde{V}L/\mathbb{C}[t^2]$.

The W -algebra $W(2, 2)$ is an extension of the Virasoro algebra and is similar to the twisted Heisenberg-Virasoro algebra (see [ADKP]). However, unlike the case of the later, the action of $I(0)$ in $W(2, 2)$ is not simisimple, so its representation theory is very different from that of the twisted Heisenberg-Virasoro algebra in a fundamental way.

The $W(2, 2)$ and its highest weight modules enter the picture naturally during our discussion on $L(1/2, 0) \otimes L(1/2, 0)$. The W -algebra $W(2, 2)$ is an extension of the Virasoro algebra and also has a very good highest weight module theory (see Section 2). Its highest weight modules produce a new class of vertex operator algebras. Contrast to the Virasoro algebra case, this class of vertex operator algebras are always irrational.

The present paper is devoted to determining all irreducible weight modules with finite dimensional weight spaces over \mathcal{L} from the motivations in [LZ] and [LJ]. More precisely we prove that there are two different classes of them. One class is formed by simple modules of intermediate series, whose weight spaces are all 1-dimensional; the other class consists of the highest(or lowest) weight modules.

The paper is arranged as follows. In Section 2, we recall some notations and collect known facts about irreducible, indecomposable modules over the classical Virasoro algebra. In Section 3, we determine all irreducible (indecomposable) weight modules of intermediate series over \mathcal{L} , i.e., irreducible (indecomposable) weight modules with all 1-dimensional weight spaces. In Section 4, we determine all irreducible uniformly bounded weight modules over \mathcal{L} which turn out to be modules of intermediate series. In Section 5, we obtain the main result of this paper: the classification of irreducible weight \mathcal{L} -modules with finite dimensional weight space. As we mentioned, they are irreducible highest, lowest weight modules, or irreducible modules of the intermediate series.

2 Basics

In this section, we collect some known facts for later use.

For any $e \in \mathbb{C}$, it is clear that

$$[x_n + neI(n), x_m + meI(m)] = (m - n)(x_{n+m} + (m + n)eI(n + m)), \quad \forall n \neq -m,$$

$$[x_n + neI(n), x_{-n} + (-n)eI(-n)] = -2nx_0 + \frac{n^3 - n}{12}C.$$

So $\{x_n + enI(n), C | n \in \mathbb{Z}\}$ spans a subalgebra $\text{Vir}[e]$ which is isomorphic to the classical Virasoro algebra. In many cases, we shall simply write $\text{Vir}[0]$ as Vir .

Introduce a \mathbb{Z} -grading on \mathcal{L} by defining the degrees: $\deg x_n = \deg I(n) = n$ and $\deg C = 0$. Set

$$\mathcal{L}_+ = \sum_{n \geq 0} (\mathbb{C}x_n + \mathbb{C}I(n)), \quad \mathcal{L}_- = \sum_{n \leq 0} (\mathbb{C}x_n + \mathbb{C}I(n)),$$

and

$$\mathcal{L}_0 = \mathbb{C}x_0 + \mathbb{C}I(0) + \mathbb{C}C.$$

An \mathcal{L} -module V is called a weight module if V is the sum of all its weight spaces. For a weight module V we define

$$\text{Supp}(V) := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\},$$

which is generally called the weight set (or the support) of V .

A nontrivial weight \mathcal{L} -module V is called a weight module of intermediate series if V is indecomposable and any weight spaces of V is one dimensional.

A weight \mathcal{L} -module V is called a highest (resp. lowest) weight module with highest weight (resp. lowest weight) $\lambda \in \mathbb{C}$, if there exists a nonzero weight vector $v \in V_\lambda$ such that

- 1) V is generated by v as \mathcal{L} -module;
- 2) $\mathcal{L}_+ v = 0$ (resp. $\mathcal{L}_- v = 0$).

Remark. For a highest (lowest) vector v we always suppose that $I_0 v = c_0 v$ for some $c_0 \in \mathbb{C}$ although the action of I_0 is not semisimple.

Obviously, if M is an irreducible weight \mathcal{L} -module, then there exists $\lambda \in \mathbb{C}$ such that $\text{Supp}(M) \subset \lambda + \mathbb{Z}$. So M is a \mathbb{Z} -graded module.

If, in addition, all weight spaces M_λ of a weight \mathcal{L} -module M are finite dimensional, the module is called a *Harish-Chandra module*. Clearly a highest (lowest) weight module is a Harish-Chandra module.

Let $U := U(\mathcal{L})$ be the universal enveloping algebra of \mathcal{L} . For any $\lambda, c \in \mathbb{C}$, let $I(\lambda, c, c_0, c_1)$ be the left ideal of U generated by the elements

$$\{x_i, I(i) \mid i \in \mathbb{N}\} \bigcup \{x_0 - \lambda \cdot 1, C - c \cdot 1, I_0 - c_0 \cdot 1, C_1 - c_1 \cdot 1\}.$$

Then the Verma module with the highest weight λ over \mathcal{L} is defined as

$$M(\lambda, c, c_0, c_1) := U/I(\lambda, c, c_0, c_1).$$

It is clear that $M(\lambda, c, c_0, c_1)$ is a highest weight module over \mathcal{L} and contains a unique maximal submodule. Let $V(\lambda, c, c_0, c_1)$ be the unique irreducible quotient of $M(\lambda, c, c_0, c_1)$.

The following result was given in [ZD].

Theorem 2.1. [ZD] *The Verma module $M(\lambda, c, c_0, c_1)$ is irreducible if and only if $\frac{m^2-1}{12}c_1 + 2c_0 \neq 0$ for any nonzero integer m .*

In the rest of this section, we recall some known facts about weight representations of the classical Virasoro algebra which can be considered as a subalgebra of \mathcal{L} :

$$\text{Vir} := \text{span}\{x_n, C \mid n \in \mathbb{Z}\}.$$

Kaplansky-Santharoubane [KS] in 1983 gave a classification of *Vir*-modules of the intermediate series. There are three families of *indecomposable modules of the intermediate series* (i.e nontrivial indecomposable weight modules with each weight space is at most one-dimensional) over the Virasoro algebra. They are *Vir*-modules "without central charges".

- (1) $\mathcal{A}_{a,b} = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i$: $x_mv_i = (a + i + bm)v_{m+i}$;
- (2) $\mathcal{A}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i$: $x_mv_i = (i + m)v_{m+i}$ if $i \neq 0$, $x_mv_0 = m(m + a)v_m$;
- (3) $\mathcal{B}(a) = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i$: $x_mv_i = iv_{m+i}$ if $i \neq -m$, $x_mv_{-m} = -m(m + a)v_0$, for some $a, b \in \mathbb{C}$.

When $a \notin \mathbb{Z}$ or $b \neq 0, 1$, it is well-known that the module $\mathcal{A}_{a,b}$ is simple. In the opposite case the module contains two simple subquotients namely the trivial module and $\mathbb{C}[t, t^{-1}]/\mathbb{C}$. Denote the nontrivial simple subquotients of $\mathcal{A}_{a,b}$, $\mathcal{A}(a)$, $\mathcal{B}(a)$ by $\mathcal{A}'_{a,b}$, $\mathcal{A}(a)'$, $\mathcal{B}(a)'$ respectively. They are all Harish-Chandra modules of the intermediate series over the Virasoro algebra. (These facts appear in many references, for example in [SZ]). We shall use T to denote the 1-dimensional trivial module, use $V'(0, 0)$ to denote the unique proper nontrivial submodule of $V(0, 1)$ (which is irreducible).

An indecomposable module V over *Vir* is said to be an *extension of the Vir-module W_1 by the Vir-module W_2* if V has a submodule isomorphic to W_1 and $V/W_1 \simeq W_2$.

Theorem 2.2. [MP2] *Let Z be an indecomposable weight Vir-module with weight spaces of dimension less than or equal to two. Then Z is one of the following:*

- 1) an intermediate series module;
- 2) an extension of $A_{\alpha,\beta}$, $A(a)$ or $B(a)$ by themselves;
- 3) an extension of A_{α,β_1} by A_{α,β_2} , where $\beta_1 - \beta_2 = 2, 3, 4, 5, 6$;
- 4) an extension of $A_{0,\beta}$ by W , where $\beta = 2, 3, 4, 5$ and W is one of: $A_{0,0}$, $A_{0,1}$, $A'_{0,0}$, $A'_{0,0} \oplus T$, T , $A(a)$ or $B(a)$;
- 5) an extension of T by $A_{0,0}$ or $A(a)$, ;
- 6) an extension of $A'_{0,0}$ by $A_{0,0}$ or $A(1)$;
- 7) an extension $A_{0,1}$ by $A_{0,0}$ or $B(0)$;
- 8) an extension of $A_{0,0}$ by $A_{0,1}$ or $A(a)$;
- 9) an extension of $A(0)$ by $A_{0,0}$;
- 10) an extension of $B(a)$ by $A_{0,1}$;
- 11) the contragredient extensions of the previous ones;

where $\alpha, \beta, \beta_1, \beta_2, a \in \mathbb{C}$.

Remark. In the above list, there are some repetitions, and not all of them can occur.

Theorem 2.3. [MP2] *There are exactly two indecomposable extensions $V = \text{span}\{v_{\alpha+i}, v'_{\alpha+i} | i \in \mathbb{Z}\}$ of $A_{\alpha,0}$ by $A_{\alpha,0}$ ($\alpha \notin \mathbb{Z}$) given by the actions $x_i v_{\alpha+n} = (\alpha+n)v_{\alpha+n+i}, \forall i \in \mathbb{Z}$ and*

- a) $x_i v'_{\alpha+n} = (\alpha+n)v'_{\alpha+n+1} - i v_{\alpha+n+1}$, for all $i, n \in \mathbb{Z}$; or
- b) $x_1 v'_{\alpha+n} = (\alpha+n)v'_{\alpha+n+1}$, $x_{-1} v'_{\alpha+n} = (\alpha+n)v'_{\alpha+n-1}$, $x_2 v'_{\alpha+n} = (\alpha+n)v'_{\alpha+n+2} + \frac{1}{(\alpha+n+2)(\alpha+n+1)} v_{\alpha+n+2}$, $x_{-2} v'_{\alpha+n} = (\alpha+n)v'_{\alpha+n-2} - \frac{1}{(\alpha+n-2)(\alpha+n-1)} v_{\alpha+n-2}$, where $\{v_{\alpha+n}, v'_{\alpha+n}\}$ forms a base of $V_{\alpha+n}$ for all $n \in \mathbb{Z}$.

We also need the following result from [MP1].

Theorem 2.4. [MP1] *Suppose that V is a weight Vir-module with finite dimensional weight spaces. Let M^+ (resp. M^-) be the maximal submodule of V with upper (resp. lower) bounded weights. If V and V^* (the contragredient module of V) do not contain trivial submodules, there exists a unique bounded submodule B such that $V = B \oplus M^+ \oplus M^-$.*

3 Irreducible weight modules with weight multiplicity one

In this section we determine all irreducible and indecomposable weight modules over \mathcal{L} with weight multiplicity one.

Let $V = \oplus \mathbb{C}v_i$ be a \mathbb{Z} -graded \mathcal{L} -module. Then the action of $I(0)$ is semisimple and we can suppose that $I(0)v_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. Moreover, by $[I(0), x_m] = mI(m)$ we have $mI(m)v_i = (\lambda_{m+i} - \lambda_i)x_mv_i$. Hence V is an irreducible (indecomposable) \mathcal{L} -module if and only if V is an irreducible (indecomposable) Vir-module.

Denoted $\mathcal{A}_{a,b,0}$, $\mathcal{A}(a,0)$, $\mathcal{B}(a,0)$ by the \mathcal{L} -module induced from the Vir-module $\mathcal{A}_{a,b}$, $\mathcal{A}(a)$, $\mathcal{B}(a)$ with the trivial actions of $I(n)$ for any $n \in \mathbb{Z}$, respectively. Moreover we also denote the nontrivial simple subquotients of $\mathcal{A}_{a,b}$, $\mathcal{A}(a)$, $\mathcal{B}(a)$ by $\mathcal{A}'_{a,b}$, $\mathcal{A}'(a)$, $\mathcal{B}'(a)$ respectively. Clearly, $\mathcal{A}'(a,0) \cong \mathcal{B}'(a,0) \cong \mathcal{A}'_{0,0,0}$. Now we shall prove that the above three kinds modules are all indecomposable modules with weight multiplicity one.

Lemma 3.1. *Let $V = \sum_{i \in \mathbb{Z}} \mathbb{C}v_i$ be an \mathbb{Z} -graded \mathcal{L} -module such that $x_nv_i = (a + i + bn)v_{i+n}$ for all $n, i \in \mathbb{Z}$ and some $a, b \in \mathbb{C}$. If $a + bn \neq 0$ for any $n \in \mathbb{Z}$, then $I(m)v_i = 0$.*

Proof. Since V is a module of the Virasoro algebra $\text{Vir} = \oplus_{m \in \mathbb{Z}} L(m)$, it is clear that $C = 0$ (cf. [SZ] for example). Suppose that $I(n)v_t = f(n, t)v_{n+t}$ for all $n, t \in \mathbb{Z}$. From $[L(n), I(m)] = (m - n)I(n + m) + \frac{1}{12}\delta_{n+m,0}(n^3 - n)C_1$, we see that

$$\begin{aligned} & f(m, t)L(n)v_{t+m} - f(m, n + t)(a + t + bn)v_{n+t+m} \\ &= (m - n)f(n + m, t)v_{n+m+t} + \frac{1}{12}\delta_{n+m,0}(n^3 - n)C_1v_t. \end{aligned} \quad (3.1)$$

In this case,

$$f(m, t)(a + t + m + bn) - f(m, n + t)(a + t + bn) = (m - n)f(n + m, t) + \frac{1}{12}\delta_{n+m,0}(n^3 - n)C_1. \quad (3.2)$$

Let $t = 0$ in (3.2), then

$$f(m, n)(a + bn) = (a + m + bn)f(m, 0) - ((m - n)f(n + m, 0) + \frac{1}{12}\delta_{n+m,0}(n^3 - n)C_1). \quad (3.3)$$

Let $m = n$ in (3.2), then

$$f(n, t)(a + (b + 1)n + t) - f(n, n + t)(a + bn + t) = 0. \quad (3.4)$$

Let $t = 0$ in (3.4), then

$$f(n, 0)(a + (b + 1)n) - f(n, n)(a + bn) = 0. \quad (3.5)$$

Let $t = -n$ in (3.4), then

$$f(n, -n)(a + bn) = f(n, 0)(a + (b - 1)n). \quad (3.6)$$

Setting $n = -m$ in (3.3), then

$$(a - mb)f(m, -m) = (a + m - bm)f(m, 0) - 2mf(0, 0) - \frac{1}{12}(m^3 - m)C_1. \quad (3.7)$$

From (3.6) and (3.7) we obtain that

$$f(m, 0) = \frac{a + bm}{a}(f(0, 0) - \frac{1}{24}(m^2 - 1)C_1), \quad m \neq 0. \quad (3.8)$$

Setting $t = n$ and $m = 0$ in (3.2), we have

$$(f(0, 2n) - f(0, n))(a + (b + 1)n) = nf(n, n). \quad (3.9)$$

Let $m = 0$ in (3.3), then

$$f(0, n)(a + bn) = (a + bn)f(0, 0) + nf(n, 0). \quad (3.10)$$

From (3.10) and (3.8) we have

$$f(0, n) = \frac{a + n}{a}f(0, 0) - \frac{1}{24a}(n^3 - n)C_1. \quad (3.11)$$

Applying (3.11) to (3.9), we have

$$f(n, n) = \frac{a + (b + 1)n}{a}(f(0, 0) - \frac{1}{24}(7n^2 - 1)C_1). \quad (3.12)$$

Combining (3.5) and (3.11) we obtain $C_1 = 0$.

Therefore

$$f(m, 0) = \frac{a + bm}{a}f(0, 0),$$

for all $m \in \mathbb{Z}$. By (3.3) we obtain that

$$f(m, n) = \frac{a + bm + n}{a}f(0, 0) = c(a + bm + n), \quad (3.13)$$

for all $m, n \in \mathbb{Z}$ and some $c \in \mathbb{F}$.

However, by $[I(m), I(n)] = 0$ we have $c = 0$. So $f(m, n) = 0$ for any $m, n \in \mathbb{Z}$.

■

Remark. In the following cases, we can also deduce that $C_1 = C_1 = 0$ as in Lemma 7.3. So in the following discussions, we always assume that $C_1 = C_1 = 0$.

Lemma 3.2. *If $V \simeq \mathcal{A}_{\alpha, \beta}$ as Vir-module, then $V \simeq \mathcal{A}_{\alpha, \beta, 0}$ as \mathcal{L} -module.*

Proof. (I.1) Suppose that $a \notin \mathbb{Z}$ and $a + bn \neq 0$ for all $n \in \mathbb{Z}$, then $f(m, n) = 0$ for all $m, n \in \mathbb{Z}$ by Lemma 3.1.

(I.2) $a \notin \mathbb{Z}$ and $a = bp$ for some $p \in \mathbb{Z} \setminus \{0\}$. So $b \neq 0, 1$. Therefore $f(m, n) = 0$ if $n + p \neq 0$ by (3.4). It follows from (3.4) that

$$f(0, -p) = 0. \quad (3.14)$$

Setting $i = -p$, $m = 0$ in (3.2) and using (3.14), we have

$$f(n, -p) = 0, \quad n \neq 0.$$

Therefore $f(m, n) = 0$ for all $m, n \in \mathbb{Z}$. ■

(I.3) $a \in \mathbb{Z}$. Since $\mathcal{A}_{a,b} \cong \mathcal{A}_{0,b}$, so we can suppose that $x_n v_i = (i + bn)v_{n+i}$ for all $n, i \in \mathbb{Z}$.

(I.3.1) $b \neq 0, 1$, then $a + bn \neq 0$ for all $n \neq 0$. So $f(m, n) = 0$ by Lemma 3.1. Therefore V is isomorphic to $\mathcal{A}_{0,b,0}$.

(I.3.2) $b = 1$. In this case (3.3) still holds and (3.2) and (3.3) becomes

$$f(m, i)(i + m + n) - f(m, n + i)(i + n) = (m - n)f(n + m, i) \quad (3.15)$$

and

$$nf(m, n) = (n - m)f(n + m, 0) + (n + m)f(m, 0) \quad (3.16)$$

Let $m = i = 0$ in (3.15) we have

$$f(0, n) = f(n, 0) + f(0, 0), \quad n \neq 0. \quad (3.17)$$

Replacing t by n and letting $n = -m$ in (3.15), we have

$$nf(m, n) = (n - m)f(m, n - m) + 2mf(0, n). \quad (3.18)$$

Replacing n by $n - m$ in (3.16), we have

$$(n - m)f(m, n - m) = (n - 2m)f(n, 0) + nf(m, 0). \quad (3.19)$$

From (3.16)-(3.19), we have

$$(n - m)f(n + m, 0) = nf(n, 0) - mf(m, 0) + 2mf(0, 0). \quad (3.20)$$

Let $n = 0$ in (3.20), we have $f(0, 0) = 0$. So (3.20) becomes

$$(n - m)f(n + m, 0) = nf(n, 0) - mf(m, 0).$$

We deduce that there exists $c, d \in \mathbb{C}$ such that

$$f(n, 0) = c + dn, \quad (3.21)$$

for all $n \in \mathbb{Z}$ and $n \neq 0$. Applying (3.20) to (3.16) and using (3.21), we have

$$f(m, n) = f(0, 0), \quad n \neq 0.$$

Therefore

$$f(m, n) = 2d + c(m + n), \quad n \neq 0.$$

By (3.15) we have $d = 0$.

Therefore

$$f(m, n) = c(m + n).$$

By $[I(m), I(n)] = 0$ we have $c = 0$. So $f(m, n) = 0$ for any $m, n \in \mathbb{Z}$.

(I.3.3) $b = 0$. (3.2) becomes

$$(i + m)f(m, i) - tf(m, n + i) = (m - n)f(m + n, i). \quad (3.22)$$

Let $i = 0$ and $n = -m$ in (3.22), then

$$f(m, 0) = f(0, 0).$$

Let $i = 0$ and $n = m$ in (3.22), then

$$f(m, 0) = 0, m \neq 0.$$

Hence

$$f(n, 0) = 0, \forall n \in \mathbb{Z}. \quad (3.23)$$

Setting $i = -n$ in (3.22) and using (3.23), we have

$$f(m, -n) = f(m + n, -n), \quad m \neq n. \quad (3.24)$$

Let $i = 1$ in (3.22) then

$$f(m, n + 1) = (m + 1)f(m, 1) - (m - n)f(m + n, 1). \quad (3.25)$$

Setting $n = -1$ in (3.25) and using (3.23), we have

$$f(m, 1) = f(m - 1, 1), m \neq -1. \quad (3.26)$$

So $f(m, 1) = c_1$ for some $c_1 \in \mathbb{F}$ and for any $m \geq -1$. $f(m, 1) = c_2$ for some $c_2 \in \mathbb{F}$ and for any $m \leq -2$.

Hence (3.25) becomes

$$f(m, n) = c_1 n, \quad m \geq -1; \quad f(m, n) = c_2 n, \quad m \leq -2. \quad (3.27)$$

By (3.24) we have $c_1 = c_2 = c$ for some $c_1 \in \mathbb{F}$.

Therefore $f(m, n) = nc$ for any $m, n \in \mathbb{Z}$. By $[I(m), I(n)] = 0$ we have $c = 0$. So $f(m, n) = 0$ for any $m, n \in \mathbb{Z}$. ■

Lemma 3.3. *If $V \cong \mathcal{A}(\alpha)$ or $\mathcal{B}(\alpha)$ as Vir-module, then $V \cong \mathcal{A}(\alpha, 0)$ or $\mathcal{B}(\alpha, 0)$.*

Proof. If $V \cong \mathcal{A}(\alpha)$, then $x_n v_i = (i + n)v_{n+i}$ if $t \neq 0$, $x_n v_0 = n(n + a)v_n$ for some $a \in \mathbb{C}$. We can deduce that $f(m, i) = 0$ for all $n, i \in \mathbb{Z}$. Therefore $V \cong \mathcal{A}(a, 0)$.

If $V \cong \mathcal{A}(\alpha)$, then $x_n v_i = tv_{n+i}$ if $t \neq -n$, $x_n v_{-n} = -n(n + a)v_0$, for some $a \in \mathbb{F}$.

We can deduce that $f(m, i) = 0$ for all $n, i \in \mathbb{Z}$. Then V is isomorphic to $\mathcal{B}(a, 0)$. ■

From Lemma 3.2 and Lemma 3.3 we have

Theorem 3.4. *Suppose that V is a nontrivial irreducible weight \mathcal{L} -module with weight multiplicity one. Then we have $V \simeq \mathcal{A}_{\alpha, \beta, 0}$ or $V \simeq \mathcal{A}'_{0,0,0}$ for some $\alpha, \beta \in \mathbb{C}$. Meanwhile, the three kinds modules listed in the before of Lemma 3.1 are all indecomposable weight \mathcal{L} -module with weight multiplicity one.*

4 Uniformly bounded irreducible weight modules

In this section, we assume that V is a uniformly bounded nontrivial irreducible weight module over \mathcal{L} . So there exists $\alpha \in \mathbb{C}$ such that $\text{Supp}(V) \subset \alpha + \mathbb{Z}$. From representation theory of Vir , we have $C = 0$ and $\dim V_{\alpha+n} = p$ for all $\alpha + n \neq 0$. If $\alpha \in \mathbb{Z}$, we also assume that $\alpha = 0$.

Consider V as a Vir -module. We have a Vir -submodule filtration

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(p)} = V,$$

where $W^{(1)}, \dots, W^{(p)}$ are Vir -submodules of V , and the quotient modules $W^{(i)}/W^{(i-1)}$ have weight multiplicity one for all nonzero weights.

Choose $v_n^1, \dots, v_n^p \in V_{\alpha+n}$ such that the images of $v_n^i + W^{(i-1)}$ form a basis of $(W^{(i)}/W^{(i-1)})_{\alpha+n}$ for all $\alpha + n \neq 0$. We may suppose that

$$x_i(v_n^1, \dots, v_n^p) = (x_i v_n^1, \dots, x_i v_n^p) = (v_{n+i}^1, \dots, v_{n+i}^p) A_{i,n},$$

where $A_{i,n}$ are upper triangular $p \times p$ matrices, and $A_{i,n}(j, j) = \alpha + n + i\beta_j$. Denote

$$I(i)(v_n^1, \dots, v_n^p) = (v_{n+i}^1, \dots, v_{n+i}^p) F_{i,n}, \quad (4.1)$$

where $F_{i,n}$ are $p \times p$ matrices.

The Lie brackets give

$$F_{i,j+n} F_{j,n} - F_{j,i+n} F_{i,n} = 0, \quad (4.2)$$

$$A_{i,j+n} A_{j,n} - A_{j,i+n} A_{i,n} = (j - i) A_{i+j,n}, \quad (4.3)$$

$$A_{i,j+n} F_{j,n} - F_{j,i+n} A_{i,n} = (j - i) F_{i+j,n} + \frac{1}{12} \delta_{i,-j} (i^3 - i) C_1 I_p, \quad (4.4)$$

where the last three formulas have the restriction $(\alpha + n)(\alpha + n + i)(\alpha + n + j)(\alpha + n + i + j) \neq 0$. We shall denote the (i, j) -entry of a matrix A by $A(i, j)$.

Lemma 4.1. *If all nontrivial irreducible sub-quotient Vir -modules of V are isomorphic to $\mathcal{A}'_{0,0,0}$, then $V \simeq \mathcal{A}'_{0,0,0}$.*

Proof. Now we can suppose that $\dim (W^{(1)})_0 \leq 1$ (If V contains a trivial submodule $\mathbb{C}v_0$, then the span $\{u_k^1 = I(k)v_0 | k \in \mathbb{Z}\}$ is a Vir -submodule, which can be chosen as $W^{(1)}$).

Claim. *The $(k, 1)$ -entry $F_{j,n}(k, 1) = 0$ for all $k \geq 2$, $n \neq 0$ and $j + n \neq 0$.*

Proof of Claim. Suppose that we have $F_{j,n}(k, 1) = 0$ for all $k \geq k_0 + 1$ ($k_0 \geq 2$), $n \neq 0$ and $j + n \neq 0$. We only need prove that $F_{j,n}(k_0, 1) = 0$ for all $n \neq 0$ and $j + n \neq 0$.

The $(k_0, 1)$ -entry of (4.4) gives

$$(n+j)F_{j,n}(k_0, 1) - F_{j,i+n}(k_0, 1)n = (j-i)F_{i+j,n}(k_0, 1), \text{ if } n \neq 0, -i, -j, -i-j. \quad (4.5)$$

Letting $j = 1$ in (4.5), we have the $(k_0, 1)$ -entry

$$(1-i)F_{i+1,n}(k_0, 1) = (n+1)F_{1,n}(k_0, 1) - nF_{1,i+n}(k_0, 1), \text{ if } n \neq 0, -1, -i, -1-i,$$

i.e.,

$$(2-j)F_{j,n}(k_0, 1) = (n+1)F_{1,n}(k_0, 1) - nF_{1,n+j-1}(k_0, 1), \text{ if } n \neq 0, -1, -j+1, -j. \quad (4.6)$$

Letting $j = 2$ in (4.6), we have

$$nF_{1,n+1}(k_0, 1) = (n+1)F_{1,n}(k_0, 1), \text{ if } n \neq 0, -1, -2.$$

Hence $F_{1,n}(k_0, 1) = nF_{1,1}(k_0, 1)$ for all $n \geq 1$, and $F_{1,n}(k_0, 1) = -\frac{n}{2}F_{1,-2}(k_0, 1)$ for all $n \leq -2$.

Moreover, by (4.5) we have

$$F_{m,n}(k_0, 1) = nF_{1,1}(k_0, 1), \quad n \geq 1, m \neq -n-1, \quad (4.a)$$

and

$$F_{m,n}(k_0, 1) = -n/2F_{1,-2}(k_0, 1), \quad n \leq -2, m \neq -n-1. \quad (4.b)$$

Suppose that $F_{1,1}(k_0, 1) \neq 0$, then $F_{n,1}(k_0, 1) \neq 0$ for all $n \geq 1$ by (4.a), then the span $u_{n+1} = I(n)v_1^1 + W^{(k_0-1)} \in W^{(k_0)}/W^{(k_0-1)}$, $n \in \mathbb{Z}$, is a nontrivial Vir-submodule. Moreover $x_mu_{n+1} = x_mI(n)v_1^1 + W^{(k_0-1)} = (n-m)I(m+n)v_1^1 + W^{(k_0-1)} = (n-m)u_{m+n+1}$, i.e. the Vir-module V has a nontrivial submodule not isomorphic to $\mathcal{A}'_{0,0}$, contradicting the assumption in the lemma.

Hence

$$F_{1,1}(k_0, 1) = 0.$$

Similarly we have

$$F_{1,-2}(k_0, 1) = 0.$$

Applying these to (4.6) we deduce that $F_{i,n}(k_0, 1) = 0$ for all $n \neq 0, -1, -i, -i+1$. Letting $i = -n-1$ in (4.5) for suitable n we deduce that $F_{i,-1}(k_0, 1) = 0$, and letting $n = -i+1$ in (4.5) for suitable i we deduce that $F_{i,-i+1}(k_0, 1) = 0$. So we have proved this Claim.

This claim ensures that $I(i)v_j^1 \in W^{(1)}$ if $j(i+j) \neq 0$. Consider the action of \mathcal{L} on $W^{(1)}$. By the same argument as in Lemma 3.3 we obtain that $F_{j,n}(1, 1) = 0$ for all $j+n \neq 0$ and $n \neq 0$.

By computing the actions $[x_i, I(j)]v_{-i-j}^1 = (j-i)I(i+j)v_{-i-j}^1$ we deduce that $\dim \sum_{j \in \mathbb{Z}} \mathbb{C}I(j)v_{-j}^1 \leq 1$. It is clear that $I(k)I(j)v_{-j}^1 = I(j)I(k)v_{-j}^1 = 0$ for all $k \neq j$, that $(j-2k)I(j)I(j)v_{-j}^1 = [x_k, I(j-k)]I(j)v_{-j}^1 = 0$ for many suitable k . Hence all weight spaces of $W = U(\mathcal{L})v_1^p$ is one dimensional. Combining with Theorem 3.4, we have proved this lemma. \square

Lemma 4.2. *If any nontrivial irreducible sub-quotient Vir-module of V is isomorphic to $\mathcal{A}_{\alpha,0}$, where $\alpha \notin \mathbb{Z}$, then we have $V \simeq \mathcal{A}_{\alpha,0,0}$.*

Proof. We use the same notations and similar discussions as in the proof of Lemma 4.1. Suppose that we have $F_{j,n}(k, 1) = 0$ for all $k \geq k_0 + 1$ ($k_0 \geq 2$), $n \neq 0$ and $j + n \neq 0$. We first want to prove that $F_{j,n}(k_0, 1) = 0$ for all n and j .

The $(k_0, 1)$ -entry of (4.4) gives

$$(\alpha + n + j)F_{j,n}(k_0, 1) - F_{j,i+n}(k_0, 1)(\alpha + n) = (j-i)F_{i+j,n}(k_0, 1). \quad (4.7)$$

Letting $j = 1$ in (4.7), we have the

$$(1-i)F_{i+1,n}(k_0, 1) = (\alpha + n + 1)F_{1,n}(k_0, 1) - (\alpha + n)F_{1,i+n}(k_0, 1),$$

i.e.,

$$(2-j)F_{j,n}(k_0, 1) = (\alpha + n + 1)F_{1,n}(k_0, 1) - (\alpha + n)F_{1,n+j-1}(k_0, 1). \quad (4.8)$$

Letting $j = 2$ in (4.8), we have $0 = (\alpha + n + 1)F_{1,n}(k_0, 1) - (\alpha + n)F_{1,n+1}(k_0, 1)$. Hence

$$F_{1,n}(k_0, 1) = \frac{\alpha + n}{\alpha} F_{1,0}(k_0, 1), \quad \forall n \in \mathbb{Z}.$$

Applying to (4.8) we obtain that

$$F_{j,n}(k_0, 1) = \frac{\alpha + n}{\alpha} F_{1,0}(k_0, 1), \quad \forall j, n \in \mathbb{Z}. \quad (4.9)$$

Suppose that $F_{1,0}(k_0, 1) \neq 0$. By re-scalaring $\{v_i^{k_0} | i \in \mathbb{Z}\}$ we may assume that

$$F_{1,0}(k_0, 1) = \alpha. \quad (4.10)$$

Case 1: $k_0 \geq 3$.

Case 1.1: $W^{(k_0)}/W^{(k_0-2)}$ is decomposable over Vir. In this case we can suitable choose $\{v_j^k | k, j \in \mathbb{Z}\}$ so that besides (4.9) we also have

$$F_{j,n}(k_0 - 1, 1) = \frac{\alpha + n}{\alpha} F_{1,0}(k_0 - 1, 1), \quad \forall j, n \in \mathbb{Z}. \quad (4.11)$$

If $F_{1,0}(k_0, 1) \neq 0$, we know that $I(1)v_0^1 \bmod W^{(k_0-2)}, v_1^{k_0} \bmod W^{(k_0-2)}$ are linearly independent, and that $I(1)v_0^1 \bmod W^{(k_0-2)}, v_1^{k_0-1} \bmod W^{(k_0-2)}$ are linearly independent. Then we can re-choose $W^{(k_0-1)}$ and $\{v_j^{k_0-1} | j \in \mathbb{Z}\}$ such that $v_1^{k_0-1} = I(1)v_0^1$. Then $F_{1,0}(k_0, 1) = 0$, furthermore $F_{j,n}(k_0, 1) = 0$, for all $j, n \in \mathbb{Z}$.

Case 1.2: $W^{(k_0)}/W^{(k_0-2)}$ is indecomposable over Vir . From Theorem 2.3, we need consider two subcases.

Case 1.2.1: $A_{i,n}(k_0-1, k_0) = -i$ for all $i, n \in \mathbb{Z}$.

Using (4.9) and (4.10), from the $(k_0-1, 1)$ -entry of (4.4), we obtain

$$(\alpha+n+j)F_{j,n}(k_0-1, 1) - i(\alpha+n) - F_{j,i+n}(k_0-1, 1)(\alpha+n) = (j-i)F_{i+j,n}(k_0-1, 1). \quad (4.12)$$

Letting $j = 1$ and $i = -1$ in (4.12), we obtain that

$$(\alpha+n+1)F_{1,n}(k_0-1, 1) = (\alpha+n)F_{1,n-1}(k_0-1, 1) + 2F_{0,n}(k_0-1, 1) - (\alpha+n). \quad (4.13)$$

Letting $i = j = 1$ and $j = i = 2$ in (4.12) respectively, we obtain that

$$F_{1,1+n}(k_0-1, 1) = \frac{\alpha+n+1}{\alpha+n}F_{1,n}(k_0-1, 1) - 1. \quad (4.14)$$

$$F_{2,2+n}(k_0-1, 1) = \frac{\alpha+n+2}{\alpha+n}F_{2,n}(k_0-1, 1) - 2. \quad (4.15)$$

From (4.13) and (4.14) we have

$$F_{1,n}(k_0-1, 1) = F_{0,n}(k_0-1, 1) - 1/2. \quad (4.16)$$

$$(\alpha+n)F_{0,n+1}(k_0-1, 1) = (\alpha+n+1)F_{0,n}(k_0-1, 1) - \alpha - n - 1/2. \quad (4.17)$$

Letting $j = 0$ and $i = 2$ in (4.12) and using (4.17), we obtain that

$$F_{2,n}(k_0-1, 1) = F_{0,n}(k_0-1, 1) + \alpha + n - 1/2. \quad (4.18)$$

Combining (4.15), (4.17) and (4.18), we have $0 = -2(\alpha+n)$, it is contradiction. Then $F_{1,0}(k_0, 1) = 0$, furthermore $F_{j,n}(k_0, 1) = 0 \forall j, n \in \mathbb{Z}$.

Case 1.2.2:

$$A_{\pm 1,n}(k_0-1, k_0) = 0, A_{\pm 2,n}(k_0-1, k_0) = \pm \frac{1}{(\alpha+n\pm 1)(\alpha+n\pm 2)}.$$

Again using (4.9) and (4.10), from the $(k_0-1, 1)$ -entry of (4.4), we obtain

$$\begin{aligned} (\alpha+n+j)F_{j,n}(k_0-1, 1) + A_{i,j+n}(k_0-1, k_0)(\alpha+n) - F_{j,i+n}(k_0-1, 1)(\alpha+n) \\ = (j-i)F_{i+j,n}(k_0-1, 1). \end{aligned} \quad (4.19)$$

Letting $j = 1$ and $i = -1$, we obtain that

$$\begin{aligned} (\alpha+n+1)F_{1,n}(k_0-1, 1) + A_{-1,1+n}(k_0-1, k_0)(\alpha+n) - F_{1,n-1}(k_0-1, 1)(\alpha+n), \\ = 2F_{0,n}(k_0-1, 1). \end{aligned} \quad (4.20)$$

i.e.

$$(\alpha+n+1)F_{1,n}(k_0-1, 1) = F_{1,n-1}(k_0-1, 1)(\alpha+n) + 2F_{0,n}(k_0-1, 1), \quad \forall j, n \in \mathbb{Z}. \quad (4.21)$$

Letting $i = j = 1$, we obtain that

$$(\alpha + n + 1)F_{1,n}(k_0 - 1, 1) - F_{1,1+n}(k_0 - 1, 1)(\alpha + n) = 0. \quad (4.22)$$

Combining (4.21) and (4.22), we deduce that

$$F_{1,n}(k_0 - 1, 1) = F_{0,n}(k_0 - 1, 1). \quad (4.23)$$

Letting $i = j = 2$, and $j = 2, i = -1$ in (4.19) respectively, we obtain that

$$(\alpha + n + 2)F_{2,n}(k_0 - 1, 1) + \frac{\alpha + n}{(\alpha + n + 3)(\alpha + n + 4)} - F_{2,2+n}(k_0 - 1, 1)(\alpha + n) = 0. \quad (4.24)$$

$$(\alpha + n + 2)F_{2,n}(k_0 - 1, 1) - F_{2,n-1}(k_0 - 1, 1)(\alpha + n) = 3F_{1,n}(k_0 - 1, 1). \quad (4.25)$$

Combining (4.24) and (4.25), we deduce that

$$F_{2,n+1}(k_0 - 1, 1) = \frac{3}{\alpha + n}F_{1,n}(k_0 - 1, 1) - \frac{\alpha + n}{3(\alpha + n + 2)(\alpha + n + 3)}. \quad (4.26)$$

Combining (4.24) and (4.26), we deduce that

$$\frac{6}{\alpha + n}F_{1,n}(k_0 - 1, 1) = \frac{\alpha + n - 1}{3(\alpha + n + 1)} - \frac{(\alpha + n)(\alpha + n - 2)}{3(\alpha + n + 3)(\alpha + n + 4)}. \quad (4.27)$$

It is contradict to (4.22). Then $F_{1,0}(k_0, 1) = 0$, furthermore $F_{j,n}(k_0, 1) = 0 \forall j, n \in \mathbb{Z}$.

Case 2: $k_0 = 2$.

Case 2.1: V is decomposable over Vir .

From the established Case 1 we may assume that $V = W^{(2)}$. Note that $A_{i,n}(1, 2) = 0$. The $(1, 1)$ -entry of (4.4) gives

$$(\alpha + j + n)F_{j,n}(1, 1) - F_{j,i+n}(1, 1)(\alpha + n) = (j - i)F_{i+j,n}(1, 1) + \frac{1}{12}\delta_{i,-j}(i^3 - i)C_1. \quad (4.28)$$

As the calculation as in Lemma 3.1, we obtain $C_1 = 0$, $F_{j,n}(1, 1) = d_{11}(\alpha + n)$ for some $d_{11} \in \mathbb{C}$ and for all $j, n \in \mathbb{Z}$.

Since $W^{(2)}$ is decomposable, by symmetry of (4.28) we have $F_{j,n}(k, l) = d_{kl}(\alpha + n)$ for some $d_{kl} \in \mathbb{C}$, for all $j, n \in \mathbb{Z}$ and $k, l = 1, 2$.

$$\text{Thus } F_{j,n} = (\alpha + n) \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}.$$

By (4.2), we have $F_{j,n} = 0$, so V is an decomposable \mathcal{L} -module and it is a contradiction.

Case 2.2: $W^{(2)}$ is indecomposable over Vir . The argument is exactly the same as in Case 1.2.2. We do not repeat it.

So far we have proved that $F_{j,n}(k, 1) = 0$ for all $n, j \in \mathbb{Z}$ and $k \neq 1$. Thus $W^{(1)}$ is an \mathcal{L} -submodule which must be V . Combining with Lemma 3.3, we have proved this lemma. \square

Denote by $(W^{(i)}/W^{(i-1)})'$ the unique nontrivial sub-quotient Vir-module of $W^{(i)}/W^{(i-1)}$. For any $x, y \in \mathbb{C}$, define $x \not\prec y$ if $y - x \notin \mathbb{N}$.

Lemma 4.3. *The module V carries a filtration $\{W^{(1)}, W^{(2)}, \dots, W^{(p)}\}$ with $(W^{(i)}/W^{(i-1)})' \simeq V'(\alpha, \beta_i)$ as Vir-modules, where $\beta_i \not\prec \beta_j$ for all $i < j$.*

Proof: We start with the filtration at the beginning of this section. The statement is true if $p = 1$, or 2 (use Theorem 2.2 and $V'(\alpha, 0) \simeq V'(\alpha, 1)$). Now we consider $p > 2$.

Suppose that we do not have $\beta_i \not\prec \beta_{i+1}$ for some i , say $\beta_{p-1} \not\prec \beta_p$, i.e., $\beta_p - \beta_{p-1} \in \mathbb{N}$. Consider $V/V^{(p-2)}$ ($p = 2$ for this module). Then we can have a submodule $X^{(p-1)} \supset V^{(p-2)}$ such that $(W^{(p)}/X^{(p-1)})' \simeq V'(\alpha, \beta_{p-1})$ and $(X^{(p-1)}/W^{(p-2)})' \simeq V'(\alpha, \beta_p)$.

By repeating this procedure several times if necessary, then we obtain the filtration required. \square

Now we are ready to classify all irreducible uniformly bounded weight modules over \mathcal{L} .

Theorem 4.4. *If V is a nontrivial irreducible uniformly bounded weight module over \mathcal{L} , then V is isomorphic to $V'(\alpha, \beta; 0)$ for some $\alpha, \beta \in \mathbb{C}$.*

Proof. By Lemma 4.3, the module V carries a filtration $\{W^{(1)}, W^{(2)}, \dots, W^{(p)}\}$ with $(W^{(i)}/W^{(i-1)})' \simeq V'(\alpha, \beta_i)$ such that

$$\beta_i \not\prec \beta_j \quad \text{for all } i < j. \quad (4.31)$$

From Lemma 4.1 and Lemma 4.2 we can suppose that $\beta_1 \neq 0, 1$.

Suppose $F_{j,n}(k, 1) = 0$ for all $j, n \in \mathbb{Z}$, $k > k_0$, where $k_0 > 1$ is a fixed integer. We need to show that $F_{j,n}(k_0, 1) = 0$ for all $j, n \in \mathbb{Z}$.

Claim 1. $F_{1,n}(k_0, 1) = 0$ except for finitely many $n \in \mathbb{Z}$.

Case 1: $\alpha \notin \mathbb{Z}$.

In this case all the restrictions for (4.2)-(4.4) disappear. Then the $(k_0, 1)$ -entry of (4.4) gives

$$(\alpha + n + j + i\beta_{k_0})F_{j,n}(k_0, 1) - F_{j,i+n}(k_0, 1)(\alpha + n + i\beta_1) = (j - i)F_{i+j,n}(k_0, 1). \quad (4.32)$$

Letting $j = 1$ we obtain

$$(1 - i)F_{i+1,n}(k_0, 1) = (\alpha + n + 1 + i\beta_{k_0})F_{1,n}(k_0, 1) - (\alpha + n + i\beta_1)F_{1,i+n}(k_0, 1). \quad (4.33)$$

Taking $i = 1$ we have

$$(\alpha + \beta_{k_0} + n + 1)F_{1,n}(k_0, 1) = (\alpha + \beta_1 + n)F_{1,n+1}(k_0, 1). \quad (4.34)$$

Letting $i = -1$ in (4.33), we have

$$2F_{0,n}(k_0, 1) = (\alpha - \beta_{k_0} + n + 1)F_{1,n}(k_0, 1) - (\alpha - \beta_1 + n)F_{1,n-1}(k_0, 1).$$

So

$$2(\alpha + n + \beta_{k_0})F_{0,n}(k_0, 1) = (\beta_1^2 - \beta_{k_0}^2 + 2\alpha + 2n + \beta_{k_0} - \beta_1)F_{1,n}(k_0, 1). \quad (4.35)$$

By using (4.32) with $i = 1, j = 0$, we deduce that

$$(\alpha + n + \beta_{k_0})F_{0,n}(k_0, 1) - F_{0,1+n}(k_0, 1)(\alpha + n + \beta_1) = -F_{1,n}(k_0, 1). \quad (4.36)$$

Combining (4.35), (4.36) and (4.34), we deduce that

$$F_{1,n}(k_0, 1) = 0 \quad (4.38)$$

except for finitely many $n \in \mathbb{Z}$

Case 2: $\alpha = 0$.

Since $W^{(1)} \simeq A_{0,\beta_1}$ and $\beta_1 \neq 0, 1$, then $\dim(W^{(1)})_0 = 1$ and we can have v_0^1 . In this case, the restrictions for (4.4) become $(n+j)(n+i+j) \neq 0$. The restrictions for (4.32)-(4.37) are $(n+j)(n+i+j) \neq 0$, $(n+1)(n+i+1) \neq 0$, $n(n+1) \neq 0$, $(n+1)(n+2) \neq 0$, $(n+1)(n+2)(n+3)(n+4) \neq 0$ and $(n+1)(n+2)(n+3)(n+4) \neq 0$, respectively. Thus we have (4.38) with exceptions $n = -1, -2, -3, -4$ and possibly one more exception $n = n_0$ (which comes from the computation of getting (4.38)). Claim 1 follows.

Claim 2. *There exists some $i_0 \in \mathbb{Z}, i_0 \neq -1, 0$, such that $I(i_0 + 1)W^{(1)} \subseteq W^{(k_0-1)}$.*

For any $j \in \mathbb{Z} \setminus \{0\}$, set

$$S_j = \{n \in \mathbb{Z} | I(j)v_n^1 \not\subseteq W^{(k_0-1)}\}.$$

By Claim 1 we know that $|S_1| < +\infty$. Choose i_0 satisfying

- (a) $i_0 \neq -1, 0$;
- (b) $i_0 > \max\{|x - y| | x, y \in S_1\} + 1$;
- (c) $-\alpha + (-1 + \beta_1)i_0 \notin S_1, i_0\beta_1 + \beta_1 - \alpha - 1 \notin S_1$ (because $\beta_1 \neq 0, 1$).

It is clear that

$$(i_0k + S_1) \cap S_1 = \emptyset, \text{ for } k \neq 0.$$

From

$$(\alpha + n - i_0 + i_0\beta)I(1)v_n^1 = I(1)x_{i_0}v_{n-i_0}^1 = (i_0 - 1)I(i_0 + 1)v_{n-i_0}^1 + x_{i_0}I(1)v_{n-i_0}^1,$$

we have

$$I(i_0 + 1)v_{n-i_0}^1 \in W^{(k_0-1)} \text{ if } n \notin S_1 \cup (i_0 + S_1). \quad (4.39)$$

Since $x_{-i_0}I(i_0 + 1)v_{n-i_0}^1 \in W^{(k_0-1)}$ for all $n \notin S_1 \cup (i_0 + S_1)$, i.e.,

$$(2i_0 + 1)I(1)v_{n-i_0}^1 + I(i_0 + 1)x_{-i_0}v_{n-i_0}^1$$

$$= (2i_0 + 1)I(1)v_{n-i_0}^1 + (\alpha + n - i_0 - i_0\beta)I(i_0 + 1)v_{n-2i_0}^1 \in W^{(k_0-1)}, \quad \forall n \notin S_1 \cup (i_0 + S_1),$$

we deduce that

$$(\alpha + n - i_0 - i_0\beta)I(i_0 + 1)v_{n-2i_0}^1 \in W^{(k_0-1)}, \quad \forall n \notin S_1 \cup (i_0 + S_1). \quad (4.40)$$

For any $n \in 2i_0 + S_1$, from (b) we have $n \notin S_1$ and $n \notin i_0 + S_1$, and from (c), we have $\alpha + n - i_0 - i_0\beta_1 \neq 0$. Applying this to (4.40) we obtain that

$$I(i_0 + 1)v_{\alpha+n}^1 \in W^{(k_0-1)}, \quad \forall n \in S_1.$$

Together with (4.39), we have

$$I(i_0 + 1)v_{n-i_0}^1 \in W^{(k_0-1)}, \quad \forall n \notin S_1. \quad (4.41)$$

From $[x_{-i_0-1}, I(i_0 + 1)] = 2(i_0 + 1)I(0)$, we have $x_{-i_0-1}I(i_0 + 1)v_{n-i_0}^1 \equiv I(i_0 + 1)x_{-i_0-1}v_{n-i_0}^1 \pmod{W^{(k_0-1)}}$. Hence

$$(\alpha + n - i_0 - (i_0 + 1)\beta_1)I(i_0 + 1)v_{n-2i_0-1}^1 \in W^{(k_0-1)}, \quad \forall n \notin S_1. \quad (4.42)$$

For $n - i_0 - 1 \in S_1$ we know that $n \notin S_1$, and by (c) we also have $(\alpha + n - i_0 - (i_0 + 1)\beta_1) \neq 0$. Thus

$$I(i_0 + 1)v_{n-i_0}^1 \in W^{(k_0-1)}, \quad \forall n \in S_1.$$

Therefore $S_{i_0+1} = \emptyset$. So we have proved this Claim.

Noting that $\{x_i, I(i_0)|i \in \mathbb{Z}\}$ generate \mathcal{L} , we know that $\mathcal{L}W^{(1)} \subset W^{(k_0-1)}$. By induction on k_0 we see that $W^{(1)}$ is an \mathcal{L} -submodule. From Theorem 3.4 we complete the proof of the theorem. \square

5 Classification of irreducible weight modules over \mathcal{L} with finite-dimensional weight spaces

Theorem 5.1. *Let V be an irreducible weight module over \mathcal{L} with all weight spaces finite-dimensional. If V is not uniformly bounded, then V is either a highest weight module or a lowest weight module.*

Proof. Consider V as a Vir-module. Let W be the smallest Vir-submodule of V such that V/W is a trivial Vir-submodule. Then W contains no trivial quotient module.

Let W' be the maximal trivial Vir-submodule of W . Then W/W' contains no trivial submodule.

Since $\dim W' + \dim V/W$ is finite, then Vir-module W/W' is not uniformly bounded. Now W/W' satisfies the conditions in Theorem 2.4. By using Theorem 2.4 to W/W' , we have some nontrivial upper bounded (or lower bounded) Vir-submodule of W/W' , say, W''/W' is a nontrivial upper bounded Vir-submodule of W/W' (i.e., the weight set of W''/W' has an upper bound).

Denote $W'' = M^+$, We know that M^+ is not uniformly bounded.

For any $j \in \mathbb{N}$, define $M^+(j) = \{v \in M^+ | I(i)v = 0 \ \forall i \geq j\}$, and let

$$M = \cup_{i \in \mathbb{N}} M^+(j).$$

It is easy to check that $x_i M^+(j) \subset M^+(j + |i|)$, i.e., M is an Vir-submodule of M^+ . Suppose that $\text{Supp}(V) \subset \alpha + \mathbb{Z}$ for some $\alpha \in \mathbb{C}$.

Claim. $M \neq 0$. Fix $\lambda_0 \in \alpha + \mathbb{Z}$. Since M^+ is not uniformly bounded, we have some $0 \neq \lambda_1 \in \text{Supp}(M^+)$ with $\lambda_1 < \lambda_0$ and $\dim (M^+)_{\lambda_1} > \dim V_{\lambda_0}$. Hence $I(\lambda_0 - \lambda_1) : (M^+)_{\lambda_1} \rightarrow V_{\lambda_0}$ is not injective. Say $v = v_{\lambda_1} \in M^+ \setminus \{0\}$ with $I(i_0)v = 0$ where $i_0 = \lambda_0 - \lambda_1 > 0$. Since $v \in M^+$, there exists $j_0 > 0$ such that $x_j v = 0$ for $j \geq j_0$. Then we have $I(i)v = 0$ for all $i \geq i_0 + j_0$. Thus $v \in M$. Claim follows.

Let Λ be the maximal weight of M , and v_Λ is one of the corresponding weight vectors. By the definition of M , there exists a nonnegative integer i_0 such that $I(i)v_\Lambda = 0$ for $i > i_0$, and $I(i_0)v_\Lambda \neq 0$ if $i_0 > 0$. If $i_0 = 0$, then v_Λ is a highest weight vector of the \mathcal{L} -module V , and we are done. So we assume that $i_0 > 0$. From $x_j I(i_0)v_\Lambda = (i_0 - j)I(i_0 + j)v_\Lambda + I(i_0)x_j v_\Lambda = I(i_0)x_j v_\Lambda = 0$ for all $j > 0$, we know that $I(i_0)v_\Lambda \neq 0$ is a highest weight vector over Vir. So $I(i_0)v_\Lambda \in M^+$. From $I(i)I(i_0)v_\Lambda = I(i_0)I(i)v_\Lambda = 0$ for all $i > i_0$, we know that $I(i_0)v_\Lambda \in M$, contradicting the choice of Λ and $i_0 > 0$. So we have proved this Theorem. \square

Combining Theorems 4.4 and Theorem 5.1 we obtain the main result of this paper:

Theorem 5.2. *If V is a nontrivial irreducible weight module over \mathcal{L} with finite dimensional weight spaces, then V is isomorphic to $V'(\alpha, \beta; 0)$ for some $\alpha, \beta \in \mathbb{C}$, or a highest or lowest weight module.*

In [LGZ], the authors proved the following theorem.

Theorem 5.3. *[LGZ] Let M be an irreducible weight L -module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim M_\lambda = \infty$. Then $\text{Supp}(M) = \lambda + \mathbb{Z}$, and for every $k \in \mathbb{Z}$, we have $\dim M_{\lambda+k} = \infty$.*

With Theorem 5.3, we classified all irreducible weight modules of the W -algebra $W(2, 2)$.

Theorem 5.4. *Let M be an irreducible weight \mathcal{L} -module. Assume that there exists $\lambda \in \mathbb{C}$ such that $0 < \dim M_\lambda < \infty$. Then M is a Harish-Chandra module. Consequently, M is either an irreducible highest or lowest weight module or an irreducible module from the intermediate series.*

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